Appendix: Consumption/Investment Problem When the Investment Opportunity Set Can be Enlarged by Information Gathering

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April 16, 2011

This article is the Appendix of the paper “Consumption/Investment Problem When the Investment Opportunity Set Can be Enlarged by Information Gathering” forthcoming in the journal Operations Research Letters. Don’t circulate this article.

Appendix

A Proof of Theorem 4.1

The Bellman equation for \( t < \tau \) is given by

\[
\beta V(x) = \max_{c \geq 0, \pi} \{(\alpha - r \mathbf{1}_m)\pi^\top V'(x) + (rx - c)V'(x) + \frac{1}{2} \pi \Sigma \pi^\top V''(x) + \frac{c^{1-\gamma}}{1-\gamma}\}, \quad (A.1)
\]

for \( x > 0 \). The maximizing \((c = C(x), \pi = \pi(x))\) satisfies

\[
V'(x) = c^{-\gamma}, \quad \pi = -\frac{V''(x)}{V''(x)}(\alpha - r \mathbf{1}_m)\Sigma^{-1} = \frac{C(x)}{\gamma C'(x)}(\alpha - r \mathbf{1}_m)\Sigma^{-1},
\]

if \( V'' < 0 \), which will be checked later.

Using an idea from Karatzas et al. [2], the Bellman equation (A.1) can be linearized by introducing a function \( X(c) := C^{-1}(x) \). That is, the Bellman equation can be transformed into the the following equation

\[
\kappa X''(c) = -\frac{\gamma(r - \beta - \kappa)}{c} X'(c) + \frac{\gamma^2}{c^2} (rX(c) - c), \quad c > 0,
\]

whose general solution using constant parameters \( B \) and \( \hat{B} \) is

\[
X(c; B, \hat{B}) = Bc^{-\gamma \lambda_+} + \hat{B} c^{-\gamma \lambda_-} + \frac{c}{K}, \quad c > 0,
\]

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where \( \lambda_+ > 0 \) and \( \lambda_- < -1 \) are two distinct solutions of the quadratic equation of \( \lambda \) given in (17). The conjectured solution is of the form

\[
X(c; \hat{B}) := \hat{B}c^{-\lambda_-} + \frac{c}{K} \quad \text{for } c > 0.
\]

The function \( X(\cdot; \hat{B}) \) is continuous, strictly increasing, and maps \([0, \infty)\) onto \([0, \infty)\) so that its inverse function, \( C(\cdot; \hat{B}) \), exists and is also continuous, strictly increasing, and maps \([0, \infty)\) onto \([0, \infty)\) if \( \hat{B} \geq 0 \).

With the notation in (20), consider the following strategy for \( z > L \) and \( \hat{B} \geq 0 \):

\[
\tau = T^z, \quad c_t = C(x_t; \hat{B}), \quad \pi_t = \frac{C(x_t; \hat{B})}{\gamma C'(x_t; \hat{B})} (\alpha - r \mathbf{1}_m) \Sigma^{-1}, \quad 0 \leq t \leq \tau, \quad (A.2)
\]

where \( C'(x; \hat{B}) \) denotes the derivative with respect to \( x \). Here, a free boundary value problem is considered since \( z \) as well as \( \hat{B} \) will be chosen appropriately.

As in equation (7.3) in Karatzas et. al. [2], the stochastic differential equation for \( \{c_t, 0 \leq t \leq \tau\} \) becomes

\[
dc_t = c_t \left( \frac{2\kappa}{\gamma} + r - K \right) dt + \frac{c_t}{\gamma} R^\top d\mathbf{w}(t),
\]

the solution of which is

\[
c_t = c_0 \exp \left[ \frac{r - \beta + \kappa}{\gamma} t + \frac{1}{\gamma} R^\top \mathbf{w}(t) \right], \quad 0 \leq t \leq \tau. \quad (A.4)
\]

Therefore \( x_t > 0 \) and \( c_t > 0 \) for all \( 0 \leq t < \tau \), that is, bankruptcy does not occur before the time \( \tau \) with the strategy \((\tau, c, \pi)\).

For \( c_0 = C(x; \hat{B}) > 0 \) (or equivalently \( x = X(c_0; \hat{B}) > 0 \)), let

\[
H(c_0) := V(\tau, c, \pi)(x) = E_x \left[ \int_0^\tau \exp (-\beta t) \frac{(c_t)^{1-\gamma}}{1-\gamma} dt + \exp (-\beta \tau) \frac{K^{-\gamma}}{1-\gamma} (x_\tau - L)^{1-\gamma} \mathbf{1}_{\{\tau < \infty\}} \right].
\]

By some calculation using (A.4), it can be shown that

\[
E_x \left[ \int_0^\infty \exp (-\beta t) \frac{c_t^{1-\gamma}}{1-\gamma} dt \right] = \frac{c_0^{1-\gamma}}{K(1-\gamma)}.
\]

From this and the fact that \( z > L \), the function \( H \) is well defined and finite on \((0, \infty)\), in particular for \( 0 < c_0 < C(x; \hat{B}) \) (or equivalently \( 0 < x < z \)). Hence as in Karatzas et. al. [2], by Theorem 13.16 of Dynkin [1] (Feynman-Kac formula), \( H \) is \( C^2 \) on \((0, C(x; \hat{B}))\) and satisfies there

\[
\beta H(c) = c \left( \frac{2\kappa}{\gamma} + r - K \right) H'(c) + \frac{c^2}{\gamma^2} \kappa_1 H''(c) + \frac{c^{1-\gamma}}{1-\gamma}. \quad (A.5)
\]
with \( \lim_{c \uparrow C(z; \hat{B})} H(c) = V_{(r, c, \pi)}(z) = \frac{K_e \gamma e^{-c} (z - L)^{1-\gamma}}{1-\gamma} \), the general solution to the equation \((A.5)\) using constant parameters \( A \) and \( \hat{A} \) is

\[
Ac^{-\gamma \rho^+} + \hat{A}c^{-\gamma \rho^-} + \frac{1}{(1-\gamma)K}c^{1-\gamma},
\]

where \( \rho_\pm := 1 + \lambda_\pm \). As in Theorem 8.8 of Karatzas et al. [2], it is shown that \( A = 0 \) so that for \( 0 < c < C(z; \hat{B}) \),

\[
H(c) = J(c; \hat{A}) := \hat{A}c^{-\gamma \rho^-} + \frac{1}{(1-\gamma)K}c^{1-\gamma}
\]

for some \( \hat{A} \) such that

\[
\lim_{c \uparrow C(z; \hat{B})} H(c) = J(C(z; \hat{B}); \hat{A}) = \frac{K_e \gamma}{1-\gamma} (z - L)^{1-\gamma}.
\]

(A.6)

It is guessed that the function \( V_{(r, c, \pi)}(x) \) should be smoothly pasted at the free boundary \( z \) and satisfy the Bellman equation \((A.1)\) on \((0, z)\) by choosing an appropriate \( \hat{B} \) and \( z \) for the strategy to be optimal.

Define a function \( G : (0, \infty) \rightarrow \mathbb{R} \) by

\[
G(z) := \frac{K_e \lambda_-}{K}(z - L) - \lambda_- z - \frac{\rho_- K_e}{(1-\gamma)K}(z - L) + \frac{\rho_-(z - L)}{1-\gamma}
\]

(A.7)

Let \( z^* \) be defined as in (18). Since \( K > 0 \), it holds that \( \lambda_- < -\frac{1}{\gamma} \), that is, \( 1 + \gamma \lambda_- < 0 \). Using this, it is easily checked that \( z^* > L \), and we get

\[
G(z^*) = 0.
\]

(A.8)

Let \( \hat{B}^* \) be defined as in (19). By calculation \( z^* - \frac{K_e}{K}(z^* - L) = L \frac{\rho_-}{1 + \gamma \lambda_-} > 0 \) so that \( \hat{B}^* > 0 \), and

\[
X(K_e(z^* - L); \hat{B}^*) = z^*, \text{ or equivalently } C(z^*; \hat{B}^*) = K_e(z^* - L).
\]

(A.9)

From now on, I proceed with \( \hat{B} = \hat{B}^* \) and \( z = z^* \). That is, I consider the strategy (21) and (22) in Theorem 4.1.

Define a function \( V(x) \) for \( x > 0 \) by

\[
V(x) := H(c^*_0) = V_{(r^*, c^*, \pi^*)}(x).
\]

(A.10)

Using \((A.6)\), \((A.9)\) and \((A.8)\), it is shown that \( \hat{A} = \frac{\lambda_-}{\rho_-} \hat{B}^* \) so that if \( 0 < c^*_0 < C(z^*; \hat{B}^*) = K_e(z^* - L) \) (or equivalently \( 0 < x < z^* \)) then

\[
H(c^*_0) = J(c^*_0; \frac{\lambda_-}{\rho_-} \hat{B}^*) = J(C(x; \hat{B}^*); \frac{\lambda_-}{\rho_-} \hat{B}^*)
\]

(A.11)
If \( c_0^* \geq C(z^*; \hat{B}^*) = K_c(z^* - L) \) (or equivalently \( x \geq z^* \)), then \( \tau^* = 0 \) and

\[
H(c_0^*) = \frac{K_{\gamma}}{1-\gamma}(x - L)^{1-\gamma}.
\]  (A.12)

That is,

\[
V(x) = J(C(x; \hat{B}^*); \frac{\lambda}{\rho_-} \hat{B}^*) \quad \text{for } 0 < x < z^*,
\]  (A.13)

and

\[
V(x) = \frac{K_{\gamma}}{1-\gamma}(x - L)^{1-\gamma} \quad \text{for } x \geq z^*.
\]  (A.14)

It is clear by (A.6) that \( V(\cdot) \) is continuous on \((0, \infty)\).

**Lemma A.1.** \( V(x) \) defined by (A.10) is strictly increasing and strictly concave for \( x > 0 \), and satisfies the HJB equation (A.1) for \( 0 < x < z^* \).

**Proof.** Note that \( V(x) \) can be written as in (A.13) and (A.14). By calculation, we have

\[
\frac{d}{dx} J(C(x; \hat{B}^*); \frac{\lambda}{\rho_-} \hat{B}^*) = \frac{J'(C(x; \hat{B}^*); \frac{\lambda}{\rho_-} \hat{B}^*)}{X'(C(x; \hat{B}^*); \hat{B}^*)} \quad \text{for } 0 < x < z^*,
\]  (A.15)

\[
> 0, \quad x > 0.
\]  (A.16)

Hence \( V(\cdot) \) is strictly increasing on \((0, z^*)\). Since \( V(\cdot) \) is continuous at \( z^* \) and \( \frac{K_{\gamma}}{1-\gamma}(x - L)^{1-\gamma} \) is a strictly increasing function of \( x \) on \((L, \infty)\), \( V(\cdot) \) is strictly increasing on \((0, \infty)\).

By (A.16), (A.9), and (A.14), the smooth pasting condition holds

\[
\lim_{x \uparrow z^*} V'(x) = (C(z^*; \hat{B}^*))^{-\gamma} = (K_2(z^* - k))^{-\gamma} = \lim_{x \downarrow z^*} V'(x).
\]  (A.18)

For \( 0 < x < z^* \),

\[
V''(x) = -\gamma(C(x; \hat{B}^*))^{-1-\gamma}C'(x; \hat{B}^*) < 0.
\]  (A.19)

This inequality and (A.18) imply that \( V(\cdot) \) is strictly concave for \( x > 0 \) since \( \frac{K_{\gamma}}{1-\gamma}(x - L)^{1-\gamma} \) is a strictly concave function of \( x \) on \((L, \infty)\). Similarly to Theorem 9.1 in Karatzas et. al. [2], it can be shown that \( V(x) \) satisfies the HJB equation (A.1) for \( 0 < x < z^* \).

By (A.18), (A.19) and the fact that \( \frac{K_{\gamma}}{1-\gamma}(x - L)^{1-\gamma} \) is infinitely differentiable function of \( x \) on \((L, \infty)\), it holds that

\[
V \in C^1(0, \infty) \cap C^2\left((0, z^*) \cup (z^*, \infty)\right).
\]  (A.20)

Furthermore, \( \lim_{x \uparrow z^*} V''(x) \) and \( \lim_{x \downarrow z^*} V''(x) \) exist and are finite since \( z^* > L \).

**Lemma A.2.** It holds that \( \frac{K_{\gamma}}{1-\gamma}(x - L)^{1-\gamma} \leq V(x) \) for \( L \leq x \leq z^* \).
Proof. Since $\frac{K\gamma}{1 - \gamma}(z^* - L)^{1 - \gamma} = V(z^*)$ it suffice to show $\frac{d}{dx} \frac{K\gamma}{1 - \gamma}(x - L)^{1 - \gamma} \geq V'(x)$ for $L < x < z^*$. Hence by (A.16) it suffice to show $K^c\gamma(x - L)^{-\gamma} \geq (C(x; \bar{B}^*)^{-\gamma}$ for $L < x < z^*$ which is equivalent to show $K^c(x - L) \leq C(x; \bar{B}^*)$ for $L < x < z^*$ which again equivalent to show $X(K^c(x - L); \bar{B}^*) \leq x$ for $L < x < z^*$. Let $\phi(x) := X(K^c(x - L); \bar{B}^*) - x$. Then $\phi(L) = -L < 0$ and $\phi(z^*) = 0$ by (A.9). Differentiating it, we get $\phi'(x) = -\gamma\lambda\bar{B}^*(K^c(x - L))^{-1 - \gamma\lambda} - \frac{K^c}{\gamma} - 1$ for $L < x < z^*$. Since $-1 - \gamma\lambda_\gamma > 0$, $\phi'(x)$ is strictly increasing so that $\phi(x)$ is convex for $L < x < z^*$. For any given $x \in (L, z^*)$, let $\lambda := \frac{x - L}{z^* - L}$. Then $0 < \lambda < 1$ and $x = \lambda L + (1 - \lambda)z$. By convexity, $\phi(x) \leq \lambda\phi(L) + (1 - \lambda)\phi(z^*) = -\lambda L < 0$. Hence $\phi(x) \leq 0$ for $L < x < z^*$. \(\square\)

Lemma A.3. $V(x)$ satisfies the following inequality for $x \geq z^*$:

$$
\beta V(x) \geq \max_{c \geq 0, \pi} \left\{ (\alpha - r1_m)\pi^\top V'(x) + (rx - c)V'(x) + \frac{1}{2}\pi\Sigma\pi^\top V''(x) + \frac{c^{1 - \gamma}}{1 - \gamma} \right\}.
$$

Proof. It is easy to check that for $x \geq z^*$

$$
\max_{c \geq 0, \pi} \left\{ (\alpha - r1_m)\pi^\top V'(x) + (rx - c)V'(x) + \frac{1}{2}\pi\Sigma\pi^\top V''(x) + \frac{c^{1 - \gamma}}{1 - \gamma} \right\} = \beta V(x) + K^c\gamma(x - L)^{-\gamma}\{rL - \frac{1}{\gamma}(x - L)(\kappa_c - \kappa)\}.
$$

Hence it suffice to show $rL - \frac{1}{\gamma}(x - L)(\kappa_c - \kappa) \leq 0$ for $x \geq z^*$. Since $\kappa_c > \kappa$ it suffice to show that $rL - \frac{1}{\gamma}(z^* - L)(\kappa_c - \kappa) \leq 0$. Using (18) it is shown that $rL - \frac{1}{\gamma}(z^* - L)(\kappa_c - \kappa) = L\frac{r\rho_\gamma - (\beta + \frac{\gamma - 1}{\gamma}\kappa)\lambda_\gamma}{1 + \gamma\lambda_\gamma}$. Since $1 + \gamma\lambda_\gamma < 0$ it remains to show $r\rho_\gamma - (\beta + \frac{\gamma - 1}{\gamma}\kappa)\lambda_\gamma \geq 0$. The definition of $K$ gives $r\rho_\gamma - (\beta + \frac{\gamma - 1}{\gamma}\kappa)\lambda_\gamma = r(1 + \lambda_\gamma) - (\beta + \frac{\gamma - 1}{\gamma}\kappa)\lambda_\gamma = r - (\beta - r + \frac{\gamma - 1}{\gamma}\kappa)\lambda_\gamma = r - (K - r)\gamma\lambda_\gamma$. Therefore, it should be proved that $r - (K - r)\gamma\lambda_\gamma \geq 0$. If $K - r \geq 0$, then the inequality clearly holds. If $K - r < 0$, then the inequality is equivalent to $\lambda_\gamma \geq \frac{r}{\gamma(K - r)}$. Since $\lambda_\gamma$ is the negative solution of the quadratic equation, $\kappa\lambda^2 - (r - \beta - \kappa)\lambda - r = 0$, it suffice to show that $\kappa\left(\frac{r}{\gamma(K - r)}\right)^2 - (r - \beta - \kappa)\frac{r}{\gamma(K - r)} - r \geq 0$, which holds since $\kappa\left(\frac{r}{\gamma(K - r)}\right)^2 - (r - \beta - \kappa)\frac{r}{\gamma(K - r)} - r = \frac{r}{\gamma(K - r)^2}\left[\kappa r - (r - \beta - \kappa)\gamma(K - r) - \gamma^2(K - r)\right] = \frac{rK\kappa}{\gamma(K - r)^2} > 0$. \(\square\)

Now Theorem 4.1 can be proved.

Fix $x > 0$. Let $(\tau, c, \pi) \in A(x)$ be arbitrary and let the corresponding wealth process be $(\xi_t)_{t=0}^\tau$. Choose $\xi$ such that $x < \xi < \infty$ and define $S_n = \inf\{t \geq 0 : \int_0^t \|\pi_s\|^2 ds = n\}$. With the notation in (20) let

$$
\tau_n := T^\xi \wedge S_n \wedge \tau \wedge n
$$

so that $\tau_n \uparrow \tau$ as $\xi \uparrow \infty$ and $n \uparrow \infty$. With a $\delta > 0$ let

$$
y_t := x_t + \delta \text{ for } 0 \leq t \leq \tau.
$$
From the fact that $V(x)$ satisfies the HJB equation (A.1) for $0 < x < z^*$ and by Lemma A.3, using generalized Itô’s rule (The rule can be applied by (A.20)), it is shown that

$$E_x \left[ \int_0^{\tau_n} \exp( - \beta t) \frac{c_i}{1 - \gamma} dt \right] \leq E_x \left[ \int_0^{\tau_n} \exp( - \beta t)[\beta V(y_t) - (\alpha - r1_m)\pi_t^T V'(y_t) - (ry_t - c_i)V'(y_t)] dt \right]$$

$$= E_x \left[ \int_0^{\tau_n} [-d(\exp( - \beta t)V(y_t)) + \exp( - \beta t)V'(y_t)\pi_t D\mathbf{w}^T(t)] \right]$$

$$- E_x\left[ \int_0^{\tau_n} r\delta \exp( - \beta t)V(y_t) dt \right]$$

$$= -E_x[\exp( - \beta \tau)V(x_{\tau_n} + \delta)] + V(x + \delta) - E_x\left[ \int_0^{\tau_n} r\delta \exp( - \beta t)V'(x_t + \delta) dt \right]$$

$$\leq -E_x[\exp( - \beta \tau)V(x_{\tau_n} + \delta)] + V(x + \delta),$$

where the last inequality comes from the fact that $V'(x) > 0$ for all $x > 0$. Since $\delta \leq x_{\tau_n} + \delta \leq \xi + \delta$, $V(x_{\tau_n} + \delta)$ is bounded so that $E_x[\exp( - \beta \tau)V(x_{\tau_n} + \delta)]$ is finite. Hence

$$V(x + \delta) \geq E_x\left[ \int_0^{\tau_n} \exp( - \beta t) \frac{c_i}{1 - \gamma} dt \right] + E_x[\exp( - \beta \tau)V(x_{\tau_n} + \delta)]$$

$$= E_x\left[ \int_0^{\tau_n} \exp( - \beta t) \frac{c_i}{1 - \gamma} dt \right] + E_x[\exp( - \beta \tau)V(x_{\tau_n} + \delta)1_{\{\tau < \infty\}}] + E_x[\exp( - \beta \tau)V(x_{\tau_n} + \delta)1_{\{\tau = \infty\}}].$$

By the monotone convergence theorem it holds that as $\xi \uparrow \infty$ and $n \uparrow \infty$

$$E_x\left[ \int_0^{\tau_n} \exp( - \beta t) \frac{c_i}{1 - \gamma} dt \right] \rightarrow E_x\left[ \int_0^{\tau} \exp( - \beta t) \frac{c_i}{1 - \gamma} dt \right].$$

Since $V(x_{\tau_n} + \delta) \geq V(\delta) > -\infty$, by Fatou’s lemma,

$$\liminf_{\xi \uparrow \infty, n \uparrow \infty} E_x[\exp( - \beta \tau)V(x_{\tau_n} + \delta)1_{\{\tau < \infty\}}] \geq E_x[\exp( - \beta \tau)V(x_{\tau} + \delta)1_{\{\tau < \infty\}}]$$

and

$$\liminf_{\xi \uparrow \infty, n \uparrow \infty} E_x[\exp( - \beta \tau)V(x_{\tau_n} + \delta)1_{\{\tau = \infty\}}] \geq V(\delta)E_x[\lim_{\xi \uparrow \infty, n \uparrow \infty} \exp( - \beta \tau)1_{\{\tau = \infty\}}] = 0.$$
Therefore,

\[
V(x + \delta) \geq E_x[\int_0^\tau \exp(-\beta t) \frac{c_1^{1-\gamma}}{1-\gamma} dt] + E_x[\exp(-\beta \tau) V(x_r + \delta) 1_{\{\tau < \infty\}}]
\]

\[
\geq E_x[\int_0^\tau \exp(-\beta t) \frac{c_1^{1-\gamma}}{1-\gamma} dt] + E_x[\exp(-\beta \tau) V(x_r) 1_{\{\tau < \infty\}}]
\]

\[
\geq E_x[\int_0^\tau \exp(-\beta t) \frac{c_1^{1-\gamma}}{1-\gamma} dt] + E_x[\exp(-\beta \tau) \frac{K c_1^{1-\gamma}}{1-\gamma}(x_r - L) 1_{\{\tau < \infty\}}]
\]

\[
= V(\tau, c, \pi; x),
\]

where the third inequality comes from Lemma A.2 and (A.14). Letting \(\delta \downarrow 0\) we get

\[
V(x) \geq V(\tau, c, \pi; x).
\]

Since \((\tau, c, \pi) \in A(x)\) is arbitrary, \(V(x) \geq V^*(x)\). Since \((\tau^*, c^*, \pi^*) \in A(x)\) and \(V(x)\) is defined by (A.10), \(V(x) \leq V^*(x)\). Consequently \(V(x) = V^*(x)\) and \((\tau^*, c^*, \pi^*)\) is optimal.

**B  Proof of Proposition 5.1**

Since \(\hat{B}^*\) is larger than zero, \(x = X(C(x; \hat{B}^*); \hat{B}^*) > \frac{C(x; \hat{B}^*)}{K}\) for all \(x > 0\). Hence \(C(x; \hat{B}^*) < Kx\) for all \(x > 0\).

**C  Proof of Proposition 5.2**

The proposition is proved by

\[
\frac{C(x; \hat{B}^*)}{C'(x; \hat{B}^*)} = C(x; \hat{B}^*)X'(C(x; \hat{B}^*); \hat{B}^*)
\]

\[
= -\gamma \lambda_\perp \hat{B}^*(C(x; \hat{B}^*))^{-\gamma \lambda_\perp} + \frac{C(x; \hat{B}^*)}{K}
\]

\[
> \hat{B}^*(C(x; \hat{B}^*))^{-\gamma \lambda_\perp} + \frac{C(x; \hat{B}^*)}{K} = x \text{ for } x > 0,
\]

where the equalities come from the definitions of the functions \(X(c; \hat{B}^*)\) and \(C(x; \hat{B}^*)\), and the inequality from the fact that \(1 + \gamma \lambda_\perp < 0\) as is mentioned in Appendix A.

**D  A solution method to the examples in Section 7**

Here, \(K_c := K_c(\kappa_c)\) defined in (15), and \(L := L(\kappa_c)\) defined by (23) are functions of a choice variable \(\kappa_c\). For a given \(x_r\), the investor chooses \(\kappa_c\) satisfying the condition
(13). However, he will not choose $\kappa_c$ such that $x_\tau = L$ (i.e., $x_\tau - L = 0$) since if he chooses such an $\kappa_c$, then $\frac{K^\gamma}{1-\gamma}(x_\tau - L)^{1-\gamma} = \frac{K^\gamma}{1-\gamma}(x_\tau - L)^{1-\gamma} < \frac{K^\gamma}{1-\gamma}(x_\tau)^{1-\gamma}$. Therefore the optimizing investor chooses $\kappa_c$ satisfying $x_\tau > L$. However, if $x_\tau \leq \theta_0$, then such a choice is impossible since $L \geq \theta_0 > 0$. Hence we can assume, without loss of generality, that $x_\tau > \theta_0$.

Consequently, at time $\tau$, the investor chooses $\kappa_c$ for a given $x_\tau > \theta_0$ to maximize $\frac{dK}{d\kappa_c}(L) = \frac{\kappa}{\gamma}(x_\tau - L)^{1-\gamma}$ subject to $x_\tau > L$ (or equivalently $\kappa < \kappa + \sqrt{\frac{x_\tau - \theta_0}{\theta_1}}$). We have

$$
\frac{d}{d\kappa_c} \left( \frac{K^\gamma}{1-\gamma}(x_\tau - L)^{1-\gamma} \right) = -\gamma K^\gamma (x_\tau - L)^{1-\gamma} + K^\gamma \left\{ \frac{d}{dL} \left( \frac{(x_\tau - L)^{1-\gamma}}{1-\gamma} \right) \right\} \frac{dL}{d\kappa_c}
$$

where, by using the fact that $\frac{dK}{d\kappa_c} = \frac{\gamma}{\gamma^2} K^\gamma (x_\tau - L)^{1-\gamma}$, $dL = 2\theta_1 (\kappa_c - \kappa)$, and $K_c = K + \frac{\gamma - 1}{2\gamma}(\kappa_c - \kappa)$,

$$Q(\kappa_c) = \frac{1}{\gamma}(x_\tau - L) - K_c \frac{dL}{d\kappa_c}
$$

$$= \frac{1}{\gamma}(x_\tau - L) - 2\theta_1 K_c (\kappa_c - \kappa)
$$

$$= -\frac{3\gamma - 2}{\gamma^2} \theta_1 (\kappa_c - \kappa)^2 - 2K \theta_1 (\kappa_c - \kappa) + \frac{x_\tau - \theta_0}{\gamma}
$$

$$= -\frac{3\gamma - 2}{\gamma^2} \theta_1 (\kappa_c - \kappa + \frac{\gamma^2 K}{3\gamma - 2})^2 + \frac{\gamma^2 K^2}{3\gamma - 2} \theta_1 + \frac{1}{\gamma}(x_\tau - \theta_0)
$$

Hence $Q(\kappa_c)$ is quadratic function of $\kappa_c$ if $\gamma \neq \frac{2}{3}$, and linear if $\gamma = \frac{2}{3}$.

To maximize $\frac{K^\gamma}{1-\gamma}(x_\tau - L)^{1-\gamma}$, I first consider the case where $\gamma > 1$ and $\kappa_c$ can be chosen in $[\kappa, \infty)$. In this case, $Q(\kappa_c) > 0$ for $\kappa < \kappa_c < \kappa^*_c(x_\tau)$, $Q(\kappa_c) < 0$ for $\kappa_c > \kappa^*_c(x_\tau)$, and $Q(\kappa^*_c(x_\tau)) = 0$, where

$$\kappa^*_c(x_\tau) = \kappa + \frac{\gamma^2}{3\gamma - 2} \left\{ \sqrt{K^2 + \frac{3\gamma - 2}{\theta_1 \gamma^3}(x_\tau - \theta_0) - K} \right\}.$$

Note that

$$L(\kappa^*_c(x_\tau)) = \theta_0 + \theta_1 (\kappa^*_c(x_\tau) - \kappa)^2
$$

$$= \theta_0 + \theta_1 \left( \frac{\gamma^2}{3\gamma - 2} \right)^2 \left\{ K^2 + \frac{3\gamma - 2}{\theta_1 \gamma^3}(x_\tau - \theta_0) + K^2 - 2K \sqrt{K^2 + \frac{3\gamma - 2}{\theta_1 \gamma^3}(x_\tau - \theta_0)} \right\}
$$

$$= \theta_0 + \frac{\gamma}{3\gamma - 2}(x_\tau - \theta_0) + 2\theta_1 \left( \frac{\gamma^2}{3\gamma - 2} \right)^2 K \left( \sqrt{K^2 + \frac{3\gamma - 2}{\theta_1 \gamma^3}(x_\tau - \theta_0)} \right)
$$

$$< \theta_0 + \frac{\gamma}{3\gamma - 2}(x_\tau - \theta_0)
$$

$$= x_\tau - 2\frac{\gamma - 1}{3\gamma - 2}(x_\tau - \theta_0)
$$

$$< x_\tau.$$
Hence $\kappa_c^*(x_r)$ satisfies the constraint $x_r > L$ and the optimizing investor chooses $\kappa_c = \kappa_c^*(x_r)$ depending on $x_r > \theta_0$. We have

$$\frac{d}{dx_r} \left( \frac{K c (\kappa_c^*(x_r))^{\gamma}}{1-\gamma} (x_r - L(\kappa_c^*(x_r)))^{1-\gamma} \right) = (K c (\kappa_c^*(x_r)))^{-\gamma-1}(x_r - L(\kappa_c^*(x_r)))^{-\gamma} \left( \frac{(K c (\kappa_c^*(x_r)))^{-\gamma} - (K c (\kappa_c^*(x_r)))^{-\gamma}}{x_r - L(\kappa_c^*(x_r)))^{-\gamma}} \right),$$

where the first equality comes from the fact that $Q(\kappa_c) = \frac{1}{\gamma} (x_r - L) - K \frac{d}{d\kappa_c}$ and the second from the fact that $Q(\kappa_c^*(x_r)) = 0$. Using this again, we have

$$\frac{d^2}{d(x_r)^2} \left( \frac{(K c (\kappa_c^*(x_r)))^{-\gamma}}{1-\gamma} (x_r - L(\kappa_c^*(x_r)))^{1-\gamma} \right) = -\gamma (K c (\kappa_c^*(x_r)))^{-\gamma}(x_r - L(\kappa_c^*(x_r)))^{-\gamma} \left( \frac{1}{\gamma} - \frac{2\theta_1 (\kappa_c^*(x_r) - \kappa)}{\gamma} \right) \frac{d^2}{d(x_r)^2} \left( \frac{(K c (\kappa_c^*(x_r)))^{-\gamma}}{1-\gamma} (x_r - L(\kappa_c^*(x_r)))^{1-\gamma} \right).$$

By some calculation, we have

$$1 - \frac{2\theta_1 (\kappa_c^*(x_r) - \kappa) \frac{d\kappa_c^*(x_r)}{dx_r}}{\gamma} = 1 - \frac{1}{3\gamma - 2} \left( 1 - \frac{K}{\sqrt{K^2 + \frac{3\gamma^2 - 2}{6\gamma^2}(x_r - \theta_0)}} \right) > 0,$$

where the inequality holds since $\gamma > 1$. Therefore, as a function of $x_r$, $\frac{(K c (\kappa_c^*(x_r)))^{-\gamma}}{1-\gamma} (x_r - L(\kappa_c^*(x_r)))^{1-\gamma}$ is increasing, concave and in $C^2(\theta_0, \infty)$.

We can also consider the case where $\gamma > 1$ and $\kappa_c$ can be chosen in $[\kappa, \kappa + M]$ for $M > 0$. In this case, the investor’s decision at time $\tau$ is $\kappa_c = \kappa_c^*(x_r)$, where $\kappa_c^*(x_r) = \max[\kappa + \frac{2^2}{3\gamma^2 - 2} \left( \sqrt{K^2 + \frac{3\gamma^2 - 2}{6\gamma^2}(x_r - \theta_0)} - K \right), \kappa + M]$. It can be checked that, as a function of $x_r$, $\frac{(K c (\kappa_c^*(x_r)))^{-\gamma}}{1-\gamma} (x_r - L(\kappa_c^*(x_r)))^{1-\gamma}$ is increasing, concave and in $C^4(\theta_0, \infty) \cap C^2((\theta_0, \xi) \cup (\xi, \infty))$ by using the fact that $Q(\kappa_c^*(x_r)) = 0$ for $\theta_0 < x_r < \xi$ and $\frac{d\kappa_c^*(x_r)}{dx_r} = 0$ if $x_r = \xi$, where $\xi$ is the point satisfying $\kappa + \frac{2^2}{3\gamma^2 - 2} \left( \sqrt{K^2 + \frac{3\gamma^2 - 2}{6\gamma^2}(x_r - \theta_0)} - K \right) = \kappa + M$.

When we consider the case where $0 < \gamma < 1$, we should consider the condition $K_c > 0$ in (15) resulting from Assumption 3.2. The condition $K_c > 0$ is equivalent to $\kappa_c < \kappa + \frac{2^2}{3\gamma - 2} K$. Therefore I assume that $\kappa_c$ can be chosen in $[\kappa, \kappa + M]$, where $0 < M < \frac{2^2}{3\gamma - 2} K$. I consider two such cases: $\gamma = \frac{2}{3}$ and $\gamma = 0.5$. First, I consider the case where $\gamma = \frac{2}{3}$ so that $0 < M < \frac{4}{3} K$. In this case, the function $Q(\kappa_c) = -2K \theta_1 (\kappa_c - \kappa) - \frac{3}{2}(x_r - \theta_0)$ is linear in $\kappa_c$. It can be shown that if $Q(\kappa_c) < 0$ which is equivalent to $x_r < \varphi := \theta_0 + \frac{2\theta_1}{3} K M$, then $Q(\kappa_c) > 0$ for $\kappa_c < \kappa_c^*(x_r)$, $Q(\kappa_c) < 0$ for $\kappa_c^*(x_r) < \kappa_c \leq \kappa + M$, $Q(\kappa_c) = 0$ at $\kappa_c = \kappa_c^*(x_r)$, and $L(\kappa_c^*(x_r)) < x_r$, where $\kappa_c^*(x_r) = \kappa + \frac{2}{3\gamma - 2}(x_r - \theta_0)$. It can also be shown that if $Q(\kappa_c + M) > 0$ (or equivalently $x_r \geq \varphi$), then $Q(\kappa_c) \geq 0$ for all $\kappa_c \in [\kappa, \kappa + M]$ and $L(\kappa_c + M) < x_r$. Therefore, the optimizing investor chooses $\kappa_c = \kappa_c^*(x_r)$ with

$$\kappa_c^*(x_r) = \begin{cases} \kappa + \frac{2\theta_1}{3\gamma - 2}(x_r - \theta_0) & \text{if } x_r < \varphi \\ \kappa + M & \text{if } x_r \geq \varphi \end{cases}.$$
It can be checked that $\kappa^*_c(x_\tau)$ is continuous at $x_\tau = \varphi$. By calculation using this and the fact that $Q(\kappa^*_c(x_\tau)) = 0$ if $x_\tau < \varphi$ and $\frac{d\kappa^*_c(x_\tau)}{dx_\tau} = 0$ if $x_\tau \geq \varphi$, it also can be checked that, as a function of $x_\tau$, $(K_c(x_\tau))^{-\gamma} - (x_\tau - L(\kappa^*_c(x_\tau)))^{1-\gamma}$ is increasing, concave, and in $C^1(\theta_0, \infty) \cap C^2((\theta_0, \varphi) \cup (\varphi, \infty))$.

Now I consider the case where $\kappa^*_c(x_\tau)$ is the solution to the equation $\bar{x}_\tau = \kappa^*_c(x_\tau)$ as a function of $x_\tau$, $\kappa^*_c(\bar{x}_\tau) = 0$ if $x_\tau < \varphi$, and $\kappa^*_c(\bar{x}_\tau) = \kappa^*_c(\bar{x}_\tau)$ if $x_\tau \geq \varphi$. It can be checked that if $x_\tau < \eta := \theta_0 + \theta_1 \left\{ \left( \frac{K}{\theta} \right)^2 - \left( \frac{K}{\theta} - M \right)^2 \right\}$, then $Q(\kappa^*_c) > 0$ for $\kappa \leq \kappa^*_c < \kappa^*_c(x_\tau)$, $Q(\kappa^*_c) < 0$ for $\kappa^*_c(x_\tau) < \kappa \leq \kappa + M$, $Q(\kappa^*_c) = 0$ at $\kappa^*_c = \kappa^*_c(x_\tau)$, and $L(\kappa^*_c(x_\tau)) < x_\tau$, where $\kappa^*_c(x_\tau) = \kappa + K - \sqrt{\left( \frac{K}{\theta} \right)^2 - \frac{1}{\theta_1}(x_\tau - \theta_0)}$. It can also be shown that if $Q(\kappa + M) \geq 0(\text{or equivalently } x_\tau \geq \eta)$, then $Q(\kappa^*_c) \geq 0$ for all $\kappa^*_c \in [\kappa, \kappa + M]$ and $L(\kappa + M) < x_\tau$. Therefore, the optimizing investor chooses $\kappa^*_c = \kappa^*_c(x_\tau)$ with

$$\kappa^*_c(x_\tau) = \left\{ \begin{array}{ll}
\frac{\kappa + K - \sqrt{\left( \frac{K}{\theta} \right)^2 - \frac{1}{\theta_1}(x_\tau - \theta_0)}}{\kappa + M} & \text{if } x_\tau < \eta
\end{array} \right.$$

It can be checked that $\kappa^*_c(x_\tau)$ is continuous at $x_\tau = \eta$. Using this and the fact that $Q(\kappa^*_c(x_\tau)) = 0$ if $x_\tau < \eta$ and $\frac{d\kappa^*_c(x_\tau)}{dx_\tau} = 0$ if $x_\tau \geq \eta$, it also can be checked that, as a function of $x_\tau$, $(K_c(x_\tau))^{-\gamma} - (x_\tau - L(\kappa^*_c(x_\tau)))^{1-\gamma}$ is increasing and in $C^1(\theta_0, \infty) \cap C^2((\theta_0, \eta) \cup (\eta, \infty))$. However, it can be shown that the concavity of it is guaranteed if $M < \frac{K}{\theta}$.

Similarly to Appendix A, it can be shown that the optimal critical wealth level $z^{**}$ is the solution to the equation $\tilde{G}(z) = 0$ with $\tilde{G}(z) = \frac{K_c(x_\tau)}{\kappa} \lambda_\eta (z - L(\kappa^*_c(z))) - \lambda_\eta z - \frac{e^{-K_c(x_\tau)}(z - L(\kappa^*_c(z)))}{(1-\gamma)K_c(x_\tau)}$. Note that $\tilde{G}(z)$ is the same as $G(z)$ in (A.7) except that the constants $\kappa_c$ and $L$ are replaced by the functions $\kappa^*_c(z)$ and $L(\kappa^*_c(z))$ respectively.

The solution $z^{**}$ can be found numerically. If the investor’s initial wealth $x$ is less than $z^{**}$, then the optimizing investor gathers information when his wealth level reaches $z^{**}$ at the level $(\kappa^*_c(z^{**}), L(\kappa^*_c(z^{**})))$. The optimal consumption/investment policy $(c^{**}, \pi^{**})$ before the time $T^{z^{**}}$ of information gathering is the same as (22) with $\kappa_c = \kappa^*_c(z^{**})$, $L = L(\kappa^*_c(z^{**}))$ (consequently $z^* = z^{**}$) in Theorem 4.1. If the investor’s initial wealth $x$ is larger than or equal to $z^{**}$, then he gathers the information immediately. The information gathering level is chosen depending on $x$ by $(\kappa^*_c(x), L(\kappa^*_c(x)))$ for $x \geq z^{**}$. 

10
References
